Spectral Structure of HD: A Brief Technical Note

Setting

Let $H \in \mathbb{R}^{n \times n}$, with $n = 2^k$, be the normalized Hadamard matrix, and let D be a diagonal matrix whose entries are independent random signs, ± 1.

Both matrices are symmetric orthogonal involutions:

$$H^2 = D^2 = I, \qquad H^T = H, \qquad D^T = D.$$

Define

$$M = HD.$$

Because both H and D are orthogonal, M is also orthogonal.

1. Iterating HD preserves norm but does not create new mixing

Since M is orthogonal,

$$\|M^k x\|_2 = \|x\|_2$$

for every k. Thus all Lyapunov exponents are zero. Repeated application of the same matrix does not produce the expanding and contracting behavior associated with a generic random matrix product.

More importantly, once D has been drawn, $M = HD$ is fixed. Applying it repeatedly therefore follows a deterministic orbit:

$$x,\ Mx,\ M^2x,\ M^3x, \ldots$$

No new randomness is introduced at each step. Spectrally, the action is simply a collection of fixed rotations and possible reflections.

This is one reason fast randomized transforms such as Fastfood and SORF use several independently randomized blocks rather than repeatedly applying one fixed HD block. The useful mixing comes from composing independently chosen transformations.

2. The spectrum of the Hadamard matrix

For the standard Hadamard construction,

$$H_n = H_2^{\otimes k}.$$

Its eigenvalues are exactly $+1$ and -1. They can be indexed by bitstrings $b \in \{0,1\}^k$, with

$$\lambda_b = (-1)^{|b|},$$

where $|b|$ is the Hamming weight of b.

Consequently, H has $n/2$ eigenvalues equal to $+1$ and $n/2$ equal to -1. In particular,

$$\det H = (-1)^{n/2}.$$

Thus H itself is not a rotation in the usual sense: it is an orthogonal reflection-like operator with two equally sized eigenspaces.

3. The spectrum of $M = HD$: principal angles

The product of two orthogonal involutions has a particularly useful geometric interpretation.

Write

$$H = 2P_V - I, \qquad D = 2P_U - I,$$

where V is the $+1$ eigenspace of H, while U is the $+1$ eigenspace of D.

The dimension of V is exactly $n/2$. The dimension of U is simply the number of $+1$ entries in D.

The product HD is governed by the **principal angles** between these two subspaces.

For each generic principal angle $\theta_j \in (0, \pi/2)$, the corresponding two-dimensional invariant subspace has the form of a rotation through

$$2\theta_j.$$

Its two eigenvalues are therefore

$$e^{+i2\theta_j}, \qquad e^{-i2\theta_j}.$$

Special cases are:

- $\theta = 0$: the two subspaces overlap exactly, giving eigenvalue $+1$.
- $\theta = \pi/2$: the corresponding directions are orthogonal, giving eigenvalue -1.

So the key geometric statement is:

> **The rotation angles of HD are twice the principal angles between the $+1$ eigenspace of the Hadamard matrix and the random sign-pattern subspace defined by D.**

Determining these angles explicitly requires understanding the relevant Hadamard eigenspaces. In the standard tensor-product formulation, their structure is closely related to the combinatorics of Hamming space and Krawtchouk polynomials. There is therefore no simple general closed-form expression for the individual angles.

4. A small worked example: $n = 4$

Consider

$$D = \mathrm{diag}(1, 1, 1, -1).$$

For the corresponding $M = HD$,

$$\mathrm{tr}(M) = -1$$

and

$$\det(M) = \det(H)\det(D) = (-1)^2(-1) = -1.$$

Because M is orthogonal, its eigenvalues lie on the unit circle and occur in conjugate pairs except for real eigenvalues ± 1.

These invariants force the spectrum to be

$$\mathrm{spec}(M) = \{+1, -1, e^{i2\pi/3}, e^{-i2\pi/3}\}.$$

Consequently,

$$M^6 = I.$$

The transformation therefore has an exactly periodic orbit structure: its nontrivial two-dimensional component is a rotation through $2\pi/3$.

This is a useful illustration of an important small-dimensional effect. In four dimensions, the algebraic constraints are strong enough to pin the rotation angle to a particularly simple value.

5. Small dimensions versus large dimensions

There is an interesting contrast between small and large n.

If a rotation angle is a rational multiple of π, then the rotation is periodic. Niven's theorem shows that if the angle itself is expressed in the appropriate rational-π form, the corresponding cosine can take only a very restricted set of rational values:

$$0, \quad \pm\tfrac{1}{2}, \quad \pm 1.$$

Small-dimensional constructions can therefore occasionally produce remarkably simple, finite-order rotations, as in the $n = 4$ example.

As n becomes large, the principal angles generally become algebraic quantities of increasing complexity. There is no reason for them to land on these special values, and exact periodicity becomes exceptional.

The resulting orbits are typically better described as **quasi-periodic**: several fixed rotation phases evolve simultaneously, and if their angles are incommensurate, the trajectory can become dense on an invariant torus.

Importantly, this is still deterministic behavior, not chaos in the usual dynamical-systems sense. The transformation remains orthogonal, so its Lyapunov exponents remain zero.

Practical takeaway

A fixed HD block is a fast, structured orthogonal transformation whose spectrum consists of rotations and possible reflections.

Repeatedly applying the same block does not progressively improve its mixing properties. Once D has been chosen, the transformation and its spectral decomposition are fixed forever.

The useful randomness therefore comes primarily from **independent randomizations**:

$$HD_1, \qquad HD_2, \qquad HD_3, \ldots$$

and from composing several such blocks.

In short:

> HD **provides a deterministic orthogonal mixing operation once its random signs have been chosen. Iteration follows a fixed spectral orbit; new mixing comes from new independent random sign patterns and their composition.**